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Number theory

Fall Semester 2013

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Quizz #4

Due Friday december 6 in recitation.

Problems:

1. Determine if the congruence $x^2 \equiv 3 \ mod \ 53$ has a solution.
2. Prove that the Diophantine equation $x^2 + 7y^2 = 138$ has no solutions. (Hint: consider modulus $m = 4$).
3. Symplify the following expression in $\mathbb{Z}[i]$, $(3 + 2i)(3 - i)^2$
4. Does $2 + i$ divide 15 ?
5. The continued fraction expansion of $\sqrt{7}$ is $[2, \overline{1, 1, 1, 4}]$. Find TWO positive solutions ($x > 0, y > 0$) to $x^2 - 7y^2 = 1$. (Hint: partial convergents).

Solution:

1. The question is whether the Legendre symbol $\left(\frac{3}{53}\right)$ is 1 or -1. Since $53 \equiv 1 \ mod \ 4$, and since $53 \equiv 2 \ mod \ 3$, we have by Quadratic Reciprocity
$$(\frac{3}{53}) = (\frac{53}{3}) = (\frac{2}{3})$$
Again by Quadratic reciprocity, the last Legengre symbol is -1 since $3 \equiv 3 \ mod \ 8$, (or observe that $2 \equiv -1 \ mod 3$ and use the formula of the course). Thus the original quadratic congruence has no solutions.
2. We know that x^2, $y^2 \in \{\bar{0}, \bar{1}\}$ modulo 4, so that $x^2 - y^2 \in \{\bar{0}, \bar{1}, \bar{3}\}$. Since modulo 4 the Diophantine equation reduces to $x^2 - y^2 \equiv 2 \ mod \ 4$, we see that there is no solution modulo 4. Hence the original Diophantine equation has no solutions.
3. $$\begin{array}{rcl} (3+2i)(3-i)^2 = (3+2i)(9-6i+i^2) & & \\ & = & (3+2i)(8-6i) \\ & = & 24 + 16i - 18i - 12i^2 \\ & = & 36 - 2i \end{array}$$
4. We would news there to be an a, $b \in \mathbb{Z}$ where $(2 + i)(a + ib) = 15$. Now $(2 + i)(a + ib) = (2a - b) + (2b + a)i$ so we need $2a - b = 15$ and $2b + a = 0$. Solving simultaneously, we get $a = 6$, $b = -3$, so the answer is yes.

5. The first few partial convergence are 2/1, 3/1, 5/2, 8/3, .. If one is checking one-by-one, one sees that $8^2 - 7 \times 3^2 = 1$, so is the smaller positive solution to $x^2 - 7y^2 = 1$. One can find the next-smallest solution by squaring $8 + 3\sqrt{7}$:

$$(8 + 3\sqrt{7})^2 = 127 + 48\sqrt{7}$$

Therefore another positive solution is $(x, y) = (127, 48)$.